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A Higgs–Chern-Simons gravity model in $2 + 1$ dimensions

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Abstract

We study a gravity model in $2 + 1$ dimensions, arising from a generalized Chern-Simons (CS) density we call a Higgs-Chern-Simons (HCS) density. This generalizes the construction of gravitational systems resulting from non-Abelian CS densities in all odd dimensions. The new HCS densities employed here are arrived at by the usual one-step descent of new Higgs–Chern-Pontryagin (HCP) densities, the latter resulting from the dimensional reduction of Chern-Pontryagin (CP) densities in some even dimension, such that in any given dimension (including even) there is an infinite tower of such models. Here, we restrict our attention to the lowest dimension, $2 + 1$, and to the simplest such model resulting from the dimensional reduction of the 3-rd CP density. We construct a black hole (BH) solution in closed form, generalizing the familiar BTZ BH. We also study the electrically charged BH solution of the same model augmented with a Maxwell term, and contrast this solution with the electrically charged BTZ BH, specifically concerning their respective thermodynamic properties.

1 Introduction

Chern-Simons gravities (CSG) derived from non-Abelian Chern-Simons (CS) densities in $2 + 1$ dimensions were proposed by Witten in Ref. [1] and they were extended to all odd dimensions by Chamseddine in Refs. [2, 3]. CSG models consist of superpositions of gravitational models of all possible higher order gravities in the given dimensions, each appearing with a precise real numerical coefficient resulting from the calculus. In this report we refer to these models, aka. Lovelock models, as p -Einstein gravities, the number p being the power of the Riemann curvature in the Lagrangian, with $p = 0$ being the cosmological constant.

The recent work [8] has proposed a new formulation of the CSG systems, which allows their construction in all, *both* odd and even dimensions. The derivation of the new-CS densities follows exactly the same method as the *usual*-CS densities in odd dimensions. The *usual* CS density results from the one-step descent of the corresponding Chern-Pontryagin (CP) density. The CP density being a total-divergence

$$\Omega_{\text{CP}} = \partial_i \Omega_i, \quad i = \mu, D; \quad \mu = 1, 2, \dots, d; \quad d = D - 1,$$

the CS density is defined as the D -th component of Ω_M , namely $\Omega_{\text{CS}} \stackrel{\text{def.}}{=} \Omega_D$.

In the new formulation, the role of the usual-CP density, which is defined in even dimensions only, is played by what we refer to as the Higgs–Chern-Pontryagin (HCP) density, described in Refs. [4, 5] and in Appendix A of Ref. [6]. These are dimensional descendents of the n th CP density in $N = 2n$ dimensions, down to residual D dimensions ($D < N = 2n$), where now D can be either odd or even. The relics of the gauge connection on the co-dimension are Higgs scalars. The remarkable property of the HCP density $\Omega_{\text{HCP}}[A, \Phi]$, which is now given in terms of both the residual gauge field A and the Higgs scalar Φ , is that like the CP density it is also a *total divergence*

$$\Omega_{\text{HCP}} = \partial_i \Omega_i, \quad i = \mu, D; \quad \mu = 1, 2, \dots, d; \quad d = D - 1.$$

Implementing now the one-step descent of the density Ω_i , one can define the corresponding new Chern-Simons density as the D -th component of Ω_i , namely $\Omega_{\text{HCS}} \stackrel{\text{def}}{=} \Omega_D$. We refer to the quantity Ω_{HCS} as the Higgs–Chern-Simons (HCS) density. A detailed definition of the HCS density is given in Refs. [4, 5, 6]. Subsequently, a similar definition for the HCS density was given in [7], but only in odd dimensions¹ and with the Higgs scalar being a complex column, not suited to the application here. In our formulation, HCS densities are given in both odd and even dimensions and in any given dimension, there is an infinite family of HCP densities in D dimensions, as also HCS densities in $d = D - 1$ dimensions, since these follow from the descent of a CP density in any dimension $N = 2n > D$.

Once the Higgs-CS (HCS) densities are calculated, they can be employed to generate gravitational theories in the same spirit as in [1, 2, 3], which could be designated as HCS gravities (HCSG). In any given dimension, there is an infinite family of such theories, each resulting from the infinite family of HCS densities.

In the passage of the HCS densities to gravitational systems in $d = D - 1$ dimensions, we see from (1) and (2) below, that the gauge group is chosen to be $SO(d)$ and the Higgs multiplet is chosen to be a D -component *isovector* of $SO(D)$ ². The cornerstone of constructing CSG models is the identification of the non-Abelian (nA) $SO(D)$ connection in $d = D - 1$ dimensions³, with the spin-connection ω_μ^{ab} and the *Vielbein* e_μ^a , ($\mu = 1, 2, 3; a=1, 2, 3$). The prescription employed in [1, 2, 3] is

$$A_\mu = -\frac{1}{2} \omega_\mu^{ab} \gamma_{ab} + \kappa e_\mu^a \gamma_{a4} \Rightarrow F_{\mu\nu} = -\frac{1}{2} \left(R_{\mu\nu}^{ab} - \kappa^2 e_{[\mu}^a e_{\nu]}^b \right) \gamma_{ab} + \kappa C_{\mu\nu}^a \gamma_{a4}, \quad (1)$$

(γ^{ab}, γ^{a4}) being the Dirac gamma matrices used in the representation of the algebra of $SO(D)$, $D = 4$. The constant κ has dimensions L^{-1} , compensating the difference of the dimensions of the spin-connection and the *Dreibein*. In (1),

$$R_{\mu\nu}^{ab} = \partial_{[\mu} \omega_{\nu]}^{ab} + (\omega_{[\mu} \omega_{\nu]})^{ab}$$

is the Riemann curvature, and $C_{\mu\nu}^a = D_{[\mu} e_{\nu]}^a$ is the torsion.

Here, in addition to (1), we posit the corresponding prescription for the Higgs scalar Φ ,

$$2^{-1} \Phi = (\phi^a \gamma_{a5} + \phi \gamma_{45}) \Rightarrow 2^{-1} D_\mu \Phi = (D_\mu \phi^a - \kappa e_\mu^a \phi) \gamma_{a5} + (\partial_\mu \phi + \kappa e_\mu^a \phi^a) \gamma_{45} \quad (2)$$

which clearly displays the *iso-four-vector* (ϕ^a, ϕ^4), that are split into the *three component* frame-vector field ϕ^a and the scalar field $\phi = \phi^4$. The covariant derivative $D_\mu \Phi$ of the Higgs scalar features the gravitational covariant derivative

$$D_\mu \phi^a = \partial_\mu \phi^a + \omega_\mu^{ab} \phi^b$$

of the frame vector field ϕ^a . Indeed, this is a vector field $\phi_\mu = e_\mu^a \phi^a$, which however has rather unusual dynamics as will be seen below. It is neither a gauge field nor a Proca field, rather, it has geometric content.

The fields (ϕ^a, ϕ) are not matter fields like, *e.g.* Maxwell or Yang-Mills, or Skyrme, *etc.* In the latter cases, the covariant derivatives are not defined by the (gravitational) spin-connection, while here they are as seen in (2). In this sense they are like spinor fields. An immediate consequence of this is that theories like the one proposed here can support solutions with torsion.

Unlike spinors however, the fields (ϕ^a, ϕ) are gravitational coordinates as they originate from the Higgs field Φ of the nA gauge theory, which itself is a (dimensional) descendent of a (higher dimensional) connection. Thus, as seen from (2), (ϕ^a, ϕ) are on the same footing as the *usual* gravitational coordinates (ω_μ^{ab}, e_μ^a). As a consequence, we would expect that the effect of (ϕ^a, ϕ) on the solutions cannot be characterised as *hair*. We expect that they support only black hole solutions and do not describe horizonless (soliton) solutions in the limit of the horizon radius vanishing, as it happens in the usual theories with hair.

¹Apart from the definition in Ref. [7] being restricted to odd dimensions, there is another important difference with our formulation. In our case the dimensional reduction is carried out on the **gauge invariant** CP density yielding the HCP density, while in [7] it is the CS density in the (higher) odd dimensions that is subjected to dimensional reduction. While the results happen to be similar, there is no guarantee they should agree since subjecting a **gauge variant** CS density to symmetry imposition as done in [7] problematic.

²These choices coincide with the representations that yield monopoles on \mathbb{R}^d described in [4]

³We do not make a choice for the signature of the space at this stage.

The gravitational models resulting from the Higgs-CS (HCS) densities via (1)-(2) are referred to as HCS gravities [8] (HCSG). In this report we have restricted our detailed study to the “simplest” HCSG model in 2+1 dimensions, namely to the HCSG model resulting from the HCS density descended from the HCP density in 6 (rather than those in 8, 10, *etc.*) dimensions. This model is an extension of the *usual* Chern-Simons gravity [1], and our solutions can be contrasted with the Banados, Teitelboim and Zanelli (BTZ) [9] black hole solution in that theory.

This paper is organized as follows. The model studied is presented in Section 2, including the equations of motion. Then in Section 3 the imposition of (radial) symmetry is carried out, for the torsion-free case where the spin-connection is restricted to the Levi-Civita connection. There are two questions that symmetry imposition must address: *a*) The consistency of the Ansatz, and *b*) the consistency of using a torsion-free Ansatz, in a theory where the torsion tensor cannot be set to zero *a priori* because of the appearance of the gravitational covariant derivative. Both these questions are addressed implicitly, in Section 3. In Section 4 we present the solutions of the system with backreacting HCSG fields.

2 A HCSG model in 2 + 1 dimensions

The Higgs–Chern-Simons density (HCS) employed here is the “simplest” example in 2 + 1 dimensions. By simplest we mean that the Higgs–Chern-Simons (HCS) density employed to construct the HCS gravity (HCSG), is the one resulting from the “simplest” Higgs–Chern-Pontryagin (HCP) density, which is defined in one dimension higher, namely in *four* dimensions. Now in *four* dimensions, HCP densities can be constructed as dimensional descendants of a CP density in $2n > 4$ dimensions, hence it is reasonable to describe the “simplest” case at hand to be the HCP density in 4 dimensions, the one that descends from the CP density in $2n = 6$ dimensions.

Since like the CP density, the HCP density is a *total divergence*, then the corresponding HCS density results from usual *one-step* descent, in this case from 4 to 3 dimensions. It may be helpful to display two such HCS densities in 2+1 dimensions, each resulting from the *one-step* descent of a HCP density th 4 dimensions, the first of which has resulted from the dimensional descent of the CP in 6 dimensions, and the second in 8 dimensions. These are

$$\Omega_{\text{HCS}}^{(3,6)} = -2\eta^2 \Omega_{\text{CS}}^{(3)} - \varepsilon^{\mu\nu\lambda} \text{Tr} \gamma_5 D_\lambda \Phi (F_{\mu\nu} \Phi + F_{\mu\nu} \Phi) , \quad (3)$$

$$\begin{aligned} \Omega_{\text{HCS}}^{(3,8)} = & 6\eta^4 \Omega_{\text{CS}}^{(3)} - \varepsilon^{\mu\nu\lambda} \text{Tr} \gamma_5 \left\{ 6\eta^2 (\Phi D_\lambda \Phi - D_\lambda \Phi \Phi) F_{\mu\nu} \right. \\ & \left. - \left[(\Phi^2 D_\lambda \Phi \Phi - \Phi D_\lambda \Phi \Phi^2) - 2(\Phi^3 D_\lambda \Phi - D_\lambda \Phi \Phi^3) \right] F_{\mu\nu} \right\} , \end{aligned} \quad (4)$$

where the leading term $\Omega_{\text{CS}}^{(3)}$ in each is the usual CS density

$$\Omega_{\text{CS}}^{(3)} = \varepsilon^{\mu\nu\lambda} \text{Tr} A_\lambda \left(F_{\mu\nu} - \frac{2}{3} A_\mu A_\nu \right) , \quad (5)$$

and where the constant η and the Higgs field Φ both have the dimensions of L^{-1} , like the gauge connection.

Our choice for the “simplest” HCS density will be (3), rather than for example (4) or ones originating from even higher dimensional CP densities. Applying the prescriptions (1) and (2), (3) yields the HCS gravitational (HCSG) model studied here,

$$\begin{aligned} \mathcal{L}_{\text{HCSG}} = & \varepsilon^{\lambda\mu\nu} \varepsilon_{abc} \left\{ 2\eta^2 \kappa \left(e_\lambda^c R_{\mu\nu}^{ab} - \frac{2}{3} \kappa^2 e_\mu^a e_\nu^b e_\lambda^c \right) \right. \\ & + \left[2(R_{\mu\nu}^{ab} - \kappa^2 e_{[\mu}^a e_{\nu]}^b) [\phi^c (\partial_\lambda \phi + \kappa e_\lambda^d \phi^d) - \phi (D_\lambda \phi^c - \kappa e_\lambda^c \phi)] \right. \\ & \left. \left. - 4\kappa \phi^a (D_\lambda \phi^b - \kappa e_\lambda^b \phi) C_{\mu\nu}^c \right] \right\} , \end{aligned} \quad (6)$$

where $C_{\mu\nu}^c$ is the torsion tensor.

In the Lagrangian (6), both the torsion $C_{\mu\nu}^c$ and the covariant derivative $D_\lambda \phi^c$ are defined in terms of the spin-connection ω_μ^{ab} , so that the variation w.r.t. to the latter will result in the appearance of the $C_{\mu\nu}^c$ in the field equations. Whether or not can the torsion be set equal to zero consistently ⁴ for a particular field configuration must be checked. This question is tackled in Section 3 where the system is subjected to static radial symmetry.

In the present work, we restrict our attentions to the Levi-Civita connection

$$\omega_\mu^{ab}[e, \partial e] = -\frac{1}{2} e_\mu^d e^{\rho a} e^{\sigma b} \partial_{[\rho} e_{\sigma]d} + \frac{1}{2} e^{\lambda[a} \partial_{[\mu} e_{\lambda]}^{b]}, \quad (7)$$

subject to checking the consistency of our Ansatz with this case.

2.1 Equations of motion

To express the equations of motion concisely, it is useful to introduce the notations

$$\mathcal{V}_c^\lambda = \varepsilon^{\lambda\mu\nu} \varepsilon_{cab} (R_{\mu\nu}^{ab} - \kappa^2 e_{[\mu}^a e_{\nu]}^b), \quad (8)$$

$$\mathcal{S} = \eta^2 - \phi^2 - |\vec{\phi}|^2. \quad (9)$$

The (modified) Einstein equations follow from the variation of (6) *w.r.t.* the Dreibein e_λ^c , yielding

$$E_c^\lambda = \kappa \mathcal{S} \mathcal{V}_c^\lambda + 4\kappa \varepsilon^{\lambda\mu\nu} \varepsilon_{cab} [D_\mu \phi^a D_\nu \phi^b - \kappa \phi (2D_\mu \phi^a - \kappa e_\mu^a \phi) e_\nu^b] = 0. \quad (10)$$

The equations resulting from the variation of the frame-vector scalar ϕ^c are

$$E_c = 2\mathcal{V}_c^\lambda (\partial_\tau \phi + \kappa \vec{e}_\lambda \cdot \vec{\phi}) + 4\varepsilon^{\lambda\mu\nu} \varepsilon_{cab} C_{\mu\nu}^a (D_\lambda \phi^b - \kappa e_\lambda^b \phi) = 0, \quad (11)$$

while the equation resulting from the variation of the scalar ϕ is

$$E = -2\mathcal{V}_c^\lambda (D_\tau \phi^c - \kappa e_\lambda^c \phi) = 0. \quad (12)$$

The torsion equations follow from the variation of (6) w.r.t. the spin connection ω_λ^{ab} , not constrained to be the Levi-Civita connection (7), yielding

$$\begin{aligned} E_{ab}^\lambda = \varepsilon^{\lambda\mu\nu} \varepsilon_{cab} \bigg\{ & -2\kappa \mathcal{S} C_{\mu\nu}^c + 8D_\mu \phi^c \partial_\nu \phi + \\ & + 8\kappa \left[(\vec{e}_\nu \cdot \vec{\phi}) D_\mu \phi^c + e_\nu^a \phi \partial_\mu \phi \right] - 8\kappa^2 \phi (\vec{\phi} \cdot \vec{e}_\nu) e_\mu^c \bigg\} = 0 \end{aligned} \quad (13)$$

Note that in (13) the curvature $R_{\mu\nu}^{ab}$ does not appear. Setting $C_{\mu\nu}^a = 0$ in (13) puts a constraint on the fields (ϕ^a, ϕ) , which for consistency should be satisfied for a given field configuration.

In (10), in (13) and in (11), the “usual” notation $|\vec{\phi}|^2 = \phi^a \phi^a$, $(\vec{e}_\tau \cdot \vec{\phi}) = e_\mu^a \phi^a$ and $(\vec{e}_\mu \cdot D_\nu \vec{\phi}) = e_\mu^a D_\nu \phi^a$ is used.

Also, in what follows we opt for Minkowskian signature and make the replacement

$$\kappa \rightarrow i\kappa, \quad h \rightarrow -ih. \quad (14)$$

As such, setting $\phi = \phi^a = 0$ results in Einstein gravity with a negative cosmological constant $\Lambda = -\kappa^2$. As found in [9] by Banados, Teitelboim and Zanelli (BTZ), this model possess black hole (BH) solutions with Anti-de Sitter asymptotics. Their study has been seminal for a better understanding of BH physics and dualities.

A natural question, which we propose to address in the following Section, is if the BTZ BHs possess generalizations within the full model (6) with excited functions ϕ, ϕ^a .

⁴Note that coupling gravity, *e.g.* to Maxwell, Yang-Mills or Skyrme/Higgs fields can consistently be considered in a torsionless framework, since the kinetic terms of these fields *do not* feature the spin-connection.

3 The solutions

3.1 The Ansatz

We convert the usual static radially symmetric metric Ansatz in $2 + 1$ dimensions

$$ds^2 = A^2(r)dr^2 + r^2d\varphi^2 - B^2(r)dt^2, \quad (15)$$

(where r, t are the radial and time coordinate, respectively, while $0 \leq \varphi < 2\pi$), to the corresponding Dreibein Ansatz as follows

$$e_r^\alpha = A n_{(1)}^\alpha, \quad e_\varphi^\alpha = r n_{(2)}^\alpha, \quad e_t^3 = B, \quad (16)$$

where

$$n_{(1)}^\alpha = \begin{pmatrix} \cos n\varphi \\ \sin n\varphi \end{pmatrix} = (\varepsilon n_{(2)})^\alpha, \quad n_{(2)}^\alpha = \begin{pmatrix} -\sin n\varphi \\ \cos n\varphi \end{pmatrix} = -(\varepsilon n_{(1)})^\alpha, \quad (17)$$

where n is an integer.

The Ansatz we use for the frame-vector field $\phi^a = (\phi^\alpha, \phi^3)$ and the scalar ϕ is

$$\phi^\alpha = f(r) n_{(1)}^\alpha, \quad \phi^3 = g(r), \quad (18)$$

$$\phi = h(r), \quad (19)$$

having used the same unit vector $n_{(1)}^\alpha$ in (18) as that given by (17).

We restrict our attention to the torsion-free case, substituting the Ansatz (16) in the Levi-Civita connection (7) and calculating the resulting components of the curvature. Using this reduced spin connection and the Ansatz (18)-(19), we calculate the reduced (covariant) derivatives of the fields (ϕ^a, ϕ) . Substituting all this in the equations (10), (11) and (12), we find that these are satisfied for $g(r) = 0$. Thus there remain four equations for the functions (A, B, f, h) , plus one constraint equation that is satisfied. We then substitute this torsion-free Ansatz in the torsion equations (13), yielding three non-trivial equations, each of which is satisfied by the ‘‘Einstein’’ equation for h' . Therefore we conclude that our torsion-free Ansatz is consistent.

We also remark that the system is described by the reduced Lagrangian resulting from the imposition of symmetry on (6),

$$\begin{aligned} 4^{-1}L_{\text{HCSG}} = & 2\eta^2\kappa \left[-\frac{BA'}{A^2} + \kappa^2 rAB \right] \\ & - \left\{ 2 \left(\frac{B'}{A^2} \right) fh' + \frac{\kappa}{A} \left[\left(\frac{BA'}{A} \right) (g^2 - h^2) - B'(f^2 + 2rhh') \right] \right. \\ & \left. + \kappa^2 [(rB' + B)fh - rB(fh' - hf')] + \kappa^3 rAB(f^2 + g^2 - 3h^2) \right\}. \end{aligned} \quad (20)$$

We observe that, indeed, (20) does not feature the derivative of the function $g(r)$, and hence one can consistently set $g(r) = 0$.

3.2 An exact solution (HBTZ BH)

In the absence of the fields (ϕ^a, ϕ) , only the equation (10) must be solved, which is of course satisfied by the vacuum BTZ solution [9]. Since the BTZ solution satisfies the equations of Chern-Simons gravity (CSG) in $2 + 1$ dimensions, and since the solution presented here results instead from the equations of the Higgs-Chern-Simons gravity (HCSG), one might refer to the new solutions as Higgs-BTZ, or simply as HBTZ black holes.

We have found that the field equations (10)-(12) possess a closed-form solution with

$$B^2(r) = \frac{1}{A^2(r)} = \kappa^2 r^2 + c_t, \quad (21)$$

$$h(r) = c_0 r, \quad f(r) = \frac{c_0}{\kappa} B(r), \quad (22)$$

where c_t, c_0 are free parameters. Equivalently, (22) can be replaced by $(f, h) \rightarrow (h, f)$.

One can see that (21) corresponds to a globally AdS_3 geometry (for $c_t > 0$) or to a BTZ black hole (for $c_t < 0$), which are solutions of the Einstein equations also for $\phi^a = \phi = 0$. Thus the fields (ϕ^a, ϕ) do not backreact on these geometries, leading to a vanishing *effective* energy-momentum tensor, *i.e.* that these solutions are *effectively* vacuum solutions⁵. This solution possesses another unusual property, namely that f and h functions diverge at infinity.

One may ask if other solutions exist apart from (21), (22) (preferably with both f and g finite in the far field). The answer seems to be negative, a strong indication in this direction coming from the study of the near horizon expansion of the solutions. The first step here is to notice that the field equations imply that the function f can be eliminated, with

$$f(r) = \frac{h'(r)}{\kappa A(r)}, \quad (23)$$

The BH solutions possess an horizon at $r = r_H > 0$, where we suppose the following approximate form of the generic solution (with $A(r) = \frac{1}{\sqrt{U(r)}}$, $B(r) = \sqrt{H(r)}$):

$$H(r) = \sum_{k \geq 0} H_{(k)}(r - r_H)^k, \quad U(r) = \sum_{k \geq 0} U_{(k)}(r - r_H)^k, \quad h(r) = \sum_{k \geq 0} h_{(k)}(r - r_H)^k, \quad (24)$$

where $H_{(k)}$, $U_{(k)}$ and $h_{(k)}$ are real numbers. After substituting (24) in the equations of motion and solving order by order, we have found that

$$H(r) = H_{(1)}(r - r_H) + \frac{H_{(1)}}{2r_h}(r - r_H)^2 + O(r - r_H)^{12}, \quad (25)$$

$$U(r) = 2\kappa^2 r_H(r - r_H) + \kappa^2(r - r_H)^2 + O(r - r_H)^{12}, \quad (26)$$

$$h(r) = \frac{h_{(0)}}{r_H}r + O(r - r_H)^{11}. \quad (27)$$

The above result has been proven up to order 11 in perturbation theory. It is likely however, that it holds to all orders, although we do not have a proof of that. This coincides with the near horizon form of the exact solution (21), (22) (note that the metric function $B(r)$ is fixed up to a constant factor $B(r) \rightarrow \lambda B(r)$). Thus we conclude that (21), (22) is likely the unique configuration compatible with a regular expansion at the horizon⁶.

A similar argument excludes the existence of particle-like solitonic solutions apart from (22) in an AdS background.

3.3 Deformed charged HBTZ black holes

The above results, namely the HBTZ solution, are consistent with the spirit of the 'no hair' theorems, which exclude the existence of BH solutions with matter fields that do not possess measured quantities subject to a Gauss Law [16], [17], [18].

The 'no hair' constraints can be circumvented for more complicated models, typically possessing gauge fields (see *e.g.* [10], [11], [12], for seminal work on hairy black holes with AdS asymptotics). Thus one can ask if the results in the previous subsection are generic and hold also in more general models, with a Lagrangian containing matter fields in addition to fields (e_μ^a, ϕ^a, ϕ) .

To address this question, we consider the simplest generalization of the model (4), with an additional Maxwell term,

$$\mathcal{L} = \mathcal{L}_{\text{HCSG}} - \frac{1}{4}F_{\mu\nu}^2, \quad (28)$$

⁵ It is interesting to note the analogy with gravitating self-dual instantonic Yang-Mills configurations with Euclidean metric see *e.g.* [13], [14], [15]. In that case the reason is that the stress tensor vanishes identically due to self-duality. In the present case, (21), (22) are solutions also for an Euclidean signature and one can easily check that the contribution of the second term in (6) to the total action is nonvanishing. Based on this analogy, one might claim that these closed form solutions are *effectively* vacuum solutions.

⁶This also agrees with our numerical results, which have failed to indicate the existence of other solutions apart from (21), (22).

with $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ the field strength tensor. It is clear that in this case there will be a nonvanishing stress tensor.

Considering the same **metric** Ansatz (15), we take a purely electric connection

$$A_\mu = (A_0, A_i) = (V(r), 0) . \quad (29)$$

The corresponding equations are easily derived; note that $V(r)$ interacts with (ϕ^a, ϕ) only via the geometry, with the existence of a first integral

$$V = \int dr \frac{Q}{r} \sqrt{AB}, \quad (30)$$

where Q is an integration constant identified with the electric charge.

For $f = g = h = 0$, the electrically charged BTZ BH [9] is recovered, with

$$B^2(r) = \frac{1}{A^2(r)} = \kappa^2 r^2 - M - \frac{Q^2}{2\kappa} \log r . \quad (31)$$

This solution possess an event horizon at $r = r_H$, where $B(r_H) = 0$.

We are interested in generalizations of this solution with nonvanishing fields (f, h) . Assuming the existence of a regular event horizon at $r = r_H > 0$, the field equations imply the following approximate form of the solutions near the horizon,

$$\begin{aligned} \frac{1}{A^2(r)} &= H_1 \kappa^2 r_H^2 (r - r_H) + \dots, \quad B^2(r) = \kappa^2 r_H^2 u_1 (r - r_H) + \dots, \\ f^2(r) &= h_1^2 H_1 r_H^2 (r - r_H) + \dots, \quad h(r) = h_0 + h_1 (r - r_H) + \dots, \end{aligned} \quad (32)$$

in terms of two free parameters (h_0, u_1) , with

$$\begin{aligned} H_1 &= \frac{2}{r_H} \left(1 + \frac{Q^2}{6(2 + h_0^2)\kappa^3 r_H^2} \left(\sqrt{1 - \frac{12h_0^2 \kappa^3 r_H^2}{Q^2}} - 1 \right) \right), \\ h_1 &= \frac{1}{2h_0 H_1 r_H^2} \left((2 + 3h_0^2)(2 - H_1 r_H) + 2h_0^2 H_1 r_H - \frac{Q^2}{\kappa^3 r_H^2} \right). \end{aligned} \quad (33)$$

Note that the condition for H_1 to be real implies the existence of a maximal value of the functions $h(r)$ at the horizon

$$h(r_H) < \frac{Q}{2\kappa r_H \sqrt{3\kappa}} . \quad (34)$$

We seek electrically charged HBTZ solutions deforming the charged BTZ BH's, with the functions (f, h) excited. We have encountered solutions that possess isolated zeros of (f, h) for some values of r , *i.e.*, displaying nodes, whose detailed study we have eschewed. Henceforth we concentrate exclusively on *nodeless, fundamental* solutions.

Generic solutions with non-standard asymptotics are found for the near horizon expansion (32). As seen in Figure 1, similar to the exact solution (22), the fields (f, h) diverge linearly as $r \rightarrow \infty$, with $(h(r), f(r)) \rightarrow c_0 r$. This divergence mixes with the logarithmic terms originating in the Maxwell part of the theory, leading to a slower decay at infinity of the functions B^2/r^2 and $A^2 r^2$, as compared to the charged BTZ case.

In addition to these generic solutions, we have found special solutions isolated in parameter space. For given (and nonzero) (r_H, Q) , a different situation is found for a particular set of near-horizon parameters (h_0, u_1) , which lead to f and h vanishing at infinity, with $c_0 = 0$ (for the data exhibited in Figure 1, this corresponds to the blue point).

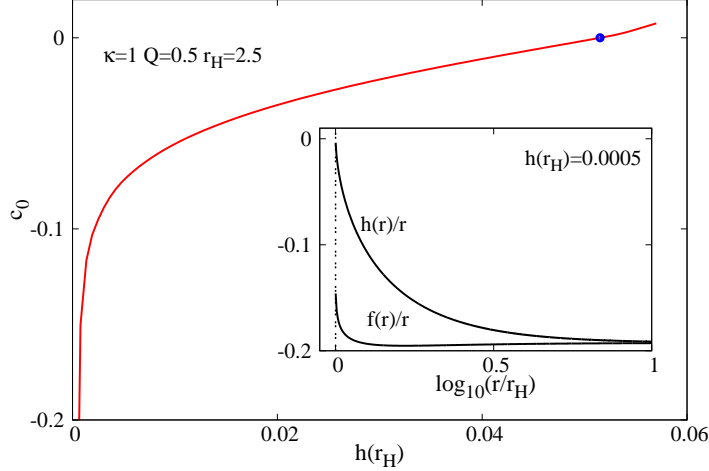


Figure 1: The parameter c_0 which enters the far field asymptotics of the functions $f(r), h(r)$ (with $(h(r), f(r)) \rightarrow c_0 r$), is shown as a function of the value of $h(r)$ at the horizon, for a set of generic solutions. The inset shows a typical profile of the functions $f(r), h(r)$.

The resulting configurations possess the following large- r asymptotic expansion:

$$A^2(r) = \frac{1}{\kappa^2 r^2} + \left(\frac{c_t - 4c_s^2}{\kappa^2} + \frac{Q^2}{2\kappa^5} \log r \right) \frac{1}{r^4} + \dots, \quad B^2(r) = \kappa^2 r^2 - c_t - \frac{Q^2}{2\kappa^5} \log r + \dots, \quad (35)$$

$$f(r) = -\frac{c_s}{r} + \dots, \quad h(r) = \frac{c_s}{r} + \dots, \quad (36)$$

in terms of two constants c_t, c_s .

This set of solutions are of particular interest and, for the remainder of this Section we shall **confine our discussions to** their basic properties. These BHs are constructed numerically by using a standard ordinary differential equation solver. In our approach, we evaluate the initial conditions (32) at $r = 10^{-5}$ for global tolerance 10^{-14} , adjusting for fixed shooting parameter and integrate the equations towards $r \rightarrow \infty$ where the far field asymptotics (35) are approached. The profiles of a typical solution is displayed in Figure 1.

The resulting solutions can be interpreted as deformations of the charged BTZ BHs, since the fields (ϕ^a, ϕ) in this case deform the geometry on and outside the horizon. To highlight the departure of the charged (deformed) HBTZ BH from the charged BTZ BH, we consider some relevant thermodynamic quantities. Their Hawking temperature T_H and event horizon area A_H are unambiguously defined, with

$$T_H = \frac{\kappa^2 r_H^2}{4\pi} \sqrt{H_1 u_1}, \quad A_H = 2\pi r_H. \quad (37)$$

For $f = h = 0$, the mass of the solutions is determined by the constant c_t in the far field asymptotics (35), while the entropy is $S = A_H/4$. However, given the direct coupling of the coupling of matter fields ϕ^a, ϕ with the curvature tensor, the definition of the BH mass and entropy for the general model (28) is not *a priori* clear, this issue requiring a separate study.

A number of basic properties of the solutions are shown in Figure 3. One can see that, similar to the charged BTZ case, there is a single branch of solutions, with both c_t and T_H increasing with A_H . Moreover, for the same horizon size, the solutions with nonzero (f, h) are warmer. Interestingly, the solutions possess a zero temperature extremal limit, which is nonsingular and features a nonvanishing horizon size. The deviation⁷ from

⁷The solutions possess a second parameter, c_s , in the far field expansion (35), which can be interpreted as a scalar 'charge', and provides another measure for the deviation of charged HBTZ BH from the (usual) charged BTZ BH. Note however, that no global charge is associated with c_s .

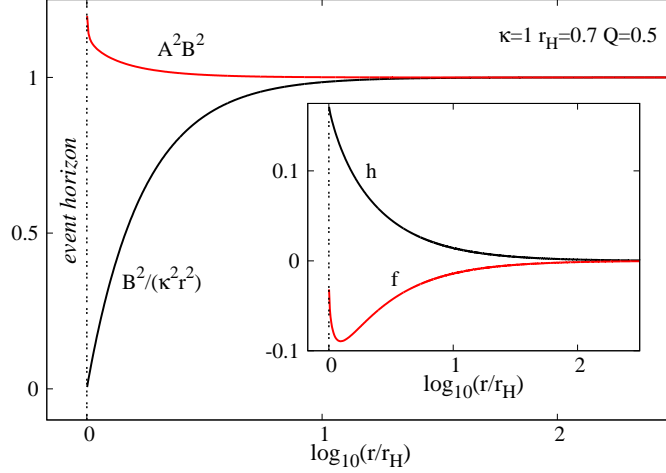


Figure 2: The profile of a typical black hole solution with nontrivial (ϕ^a, ϕ) -fields.

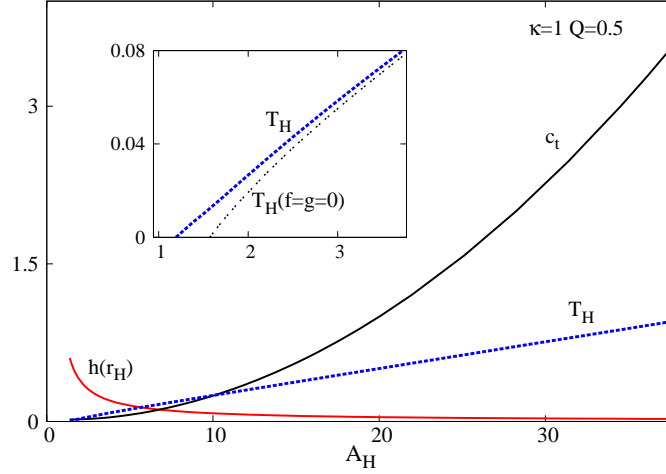


Figure 3: Relevant data is shown for a family of charged black hole solutions.

the standard charged BTZ BH is maximal, close to that limit, while the fields (ϕ^a, ϕ) trivialize for large horizon size, $(f, g) \rightarrow 0$ as $r_H \rightarrow \infty$.

4 Summary and outlook

In this paper we have studied a generalization of the familiar Chern-Simons gravity (CSG) in $2+1$ dimensions proposed in Ref. [1], where the gravitational model is described by both the Einstein-Hilbert (EH) Lagrangian with a (negative) cosmological constant. In our generalized model, in addition to the EH and cosmological terms, our Lagrangian features new terms described by a frame-vector field ϕ^a and a scalar field ϕ . The dynamical terms of the fields (ϕ^a, ϕ) are non-standard⁸.

The generalization that introduces the fields (ϕ^a, ϕ) is a result of the following. While the construction of

⁸By standard, we mean that the dynamical term of the field $\phi_\mu = e_\mu^a \phi_a$ would feature the *square* of the velocity field $\partial_\mu \phi_\nu$.

Chern-Simons gravities in all odd dimensions employs [1][2, 3] non-Abelian Chern-Simons densities, we employ instead an alternative type of generalized CS densities that we have referred to as Higgs–Chern-Simons [4, 5, 6] (HCS) densities. The HCS densities are extracted from the Higgs–Chern-Pontryagin [4] (HCP) densities, which being descendents of Chern-Pontryagin (CP) constitute an infinite family in any *even* given, including *even* spacetime dimensions. This construction is described in Ref. [8]. The present work is a first exploration of such gravitational models, and we have chosen the lowest dimension, $2 + 1$, and simplest model in this dimension, which employs the HCS density extracted from the 3-rd CP density in 6 dimensions.

The HCS gravity model we have studied, (6), features the gravitational covariant derivatives of the frame vector field ϕ^a , $a = 1, 2, 3$ in this case, so that the possibility of finding non-zero torsion solutions is not excluded. Here we have chosen to seek only torsion-free solutions. We have constructed radially symmetric solutions employing the usual metric [9] Ansatz, augmented by a suitable Ansatz for the fields (ϕ^a, ϕ) . We have verified the consistency of our Ansatz, and what is more is, that we have verified that the torsion equations resulting from the variation of (6) with respect to the spin-connection are identically satisfied by the equations resulting from the variation of (6) with respect to the metric fields, in *this* Ansatz.

We have found a closed form solution analogous to the BTZ [9] black hole, which we have referred to as a Higgs-BTZ HBTZ black hole (BH). We have a heuristic verification of the fact that this HBTZ BH is unique, and most importantly that there are no regular solutions in this model. In this respect, the HBTZ BHs cannot be considered as hairy solutions, as would have been the case if matter fields were present and the solutions persisted in the limit of vanishing horizon radius. As such, the fields (ϕ^a, ϕ) should not be seen as matter fields.

With the intention of introducing a matter field, we have extended our model to feature a Maxwell term. We have sought, and found, new solutions by adding the Maxwell field to (6). As in the case of the electrically charged BTZ BH [9], the Abelian matter does not result in hairy solutions and the solutions we find are qualitatively similar to the electrically charged BTZ BH. Unlike the latter however, these electrically charged HBTZ BHs are not given in closed form but are constructed numerically. We have found it useful to consider some thermodynamic properties of the electrically charged HBTZ BHs, by way of contrasting them to the electrically charged BTZ BHs.

As this is a preliminary exploration of such systems, there is a long list of follow-up investigations, which we list:

- Seek torsionful solutions in the model studied here.
- Analyze the present model, augmented by the planar Skyrme model in $2 + 1$ dimensions, by way of introducing a matter field. This would be an alternative to the Maxwell field considered here as a matter field. We expect that with the Skyrme matter, regular solutions in the limit of vanishing horizon radius may exist.
- Consider the HBTZ BHs in $2 + 1$ dimensions, of the next HCSG model in the hierarchy, namely (4) that results from the dimensional reduction of the 4-th CP density in 8 dimensions. We expect the resulting HBTZ BH to be qualitatively similar to the one studied here.
- Consider the HBTZ BHs in $4 + 1$ and $6 + 1$ dimensions, in the HCSG models constructed from HCS densities extracted from HCP densities descended from the 4-th and 5-th CP densities given in Ref. [8]. This would throw light on some general properties of HCSG BHs in all odd dimensions, analogous to the property common to CSG BHs in all odd dimensions, namely that the simplest solutions possesses a generic form with $\frac{1}{g_{rr}} = -g_{tt} = \kappa^2 r^2 + c_t$, starting with the BTZ BH in $2 + 1$ dimensions.
- Consider the HBTZ BHs in $3 + 1$ dimensions, in the HCSG models constructed from HCS densities extracted from HCP densities descended from the 3-rd and 4-th CP densities [8]. This would be a novel result in that Chern-Simons gravitational BHs in *even* dimensional spacetimes may also be of some interest.

In addition to the above systematic follow-ups, it is possible to consider a more formal possibility. This is the analogue of augmenting a given gravitational Lagrangian, with a gravitational Chern-Simons (GCS) term, most familiar known since a long time from Ref. [19] where the usual Einstein Lagrangian was augmented

by the GCS term in $2 + 1$ dimensions. In the present case for example, one might consider the gravitational Higgs–Chern–Simons term (GHCS), since this one features the same field multiplets. (e_μ^a, ϕ^a, ϕ) as the HCSG model studied. (The definitions of GCHS terms, along with the usual GCS densities, is given in Ref. [8]. These are derived from a *modified version*⁹ of the HCS densities rather than from the CS densities in the *usual* case.)

The simplest such GHCS term in $2 + 1$ dimensions, which has the same dimensions as (6), is

$$\hat{\Omega}_{\text{GHCS}}^{(3,6)} = -2\eta^2 \hat{\Omega}_{\text{GCS}}^{(3)} + 4\varepsilon^{\lambda\mu\nu} \phi^a R_{\mu\nu}^{ab} D_\lambda \phi^b, \quad (38)$$

$$\hat{\Omega}_{\text{GCS}}^{(3)} = -\frac{1}{2} \varepsilon^{\lambda\mu\nu} \omega_\lambda^{ab} \left[R_{\mu\nu}^{ab} - \frac{2}{3} (\omega_\mu \omega_\nu)^{ab} \right], \quad (39)$$

$\hat{\Omega}_{\text{GCS}}^{(3)}$ in (39) being the usual GCS term used in Ref. [19, 20]. Note that the GHCS density (38) features only the frame-vector field ϕ^a and not the scalar $\phi = \phi^4$, while the latter is present in the HCSG model (6).

Perhaps more interesting are the first two GHCS densities in $3 + 1$ dimensions, each derived from the HCS densities extracted, respectively, from HCP density densities in 6 and in 8 dimensions. These are

$$\hat{\Omega}_{\text{GHCS}}^{(4,6)} = -\frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{ab} R_{\rho\sigma}^{ab} \phi \quad (40)$$

$$\hat{\Omega}_{\text{GHCS}}^{(4,8)} = -\varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{ab} \left\{ \left[\frac{1}{8} \left(1 - \frac{1}{3} (|\phi^a|^2 + \phi^2) \right) R_{\rho\sigma}^{ab} + \frac{1}{3} \phi_{\rho\sigma}^{ab} \right] \phi + \frac{4}{3} \phi^a D_\rho \phi^b \partial_\sigma \phi \right\} \quad (41)$$

where the abbreviated notation

$$\phi_{\mu\nu}^{ab} = D_{[\mu} \phi^a D_{\nu]} \phi^b, \quad a = 1, 2, 3, 4; \quad \text{and} \quad \phi = \phi^5.$$

The reason for displaying (41) in addition to (40) is that the former does not feature any dynamical terms for (ϕ^a, ϕ) , while the latter does. The GHCS densities (40) and (41) can be viewed as alternatives to the GHCS density proposed in Ref. [21].

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References

- [1] E. Witten, “(2+1)-dimensional gravity as an exactly soluble system,” Nucl. Phys. B **311** (1988) 46.
- [2] A. H. Chamseddine, “Topological gauge theory of gravity in five-dimensions and all odd dimensions,” Phys. Lett. B **233** (1989) 291.
- [3] A. H. Chamseddine, “Topological gravity and supergravity in various dimensions,” Nucl. Phys. B **346** (1990) 213.
- [4] D. H. Tchrakian, “Notes on Yang–Mills–Higgs monopoles and dyons on \mathbb{R}^D , and Chern–Simons–Higgs solitons on \mathbb{R}^{D-2} : Dimensional reduction of Chern–Pontryagin densities,” J. Phys. A **44** (2011) 343001 [arXiv:1009.3790 [hep-th]].
- [5] E. Radu and D. H. Tchrakian, “New Chern–Simons densities in both odd and even dimensions,” arXiv:1101.5068 [hep-th].
- [6] D. H. Tchrakian, “Higgs-and Skyrme-Chern–Simons densities in all dimensions,” J. Phys. A **48** (2015) no.37, 375401 [arXiv:1505.05344 [hep-th]].

⁹In odd dimensions, where the HCS term is extracted from a HCP term in even dimensions, the chiral matrix under the Trace of the latter must be removed *by hand*. This *ad hoc* step is taken so that the leading term in the GHCS density is the GCS density in the given dimension. This question does not arise in even dimensional spacetimes. .

- [7] R. J. Szabo and O. Valdivia, “Covariant quiver gauge theories,” *JHEP* **1406** (2014) 144, [arXiv:1404.4319 [hep-th]].
- [8] D. H. Tchrakian, “Chern-Simons gravities (CSG) and gravitational Chern-Simons (GCS) densities in all dimensions,” arXiv:1712.05190 [gr-qc].
- [9] M. Banados, C. Teitelboim and J. Zanelli, “The black hole in three-dimensional space-time,” *Phys. Rev. Lett.* **69** (1992) 1849 [hep-th/9204099].
- [10] S. S. Gubser, “Breaking an Abelian gauge symmetry near a black hole horizon,” *Phys. Rev. D* **78** (2008) 065034 [arXiv:0801.2977 [hep-th]].
- [11] S. S. Gubser, “Colorful horizons with charge in anti-de Sitter space,” *Phys. Rev. Lett.* **101** (2008) 191601 [arXiv:0803.3483 [hep-th]].
- [12] E. Winstanley, “Existence of stable hairy black holes in SU(2) Einstein-Yang-Mills theory with a negative cosmological constant,” *Class. Quant. Grav.* **16** (1999) 1963 [gr-qc/9812064].
- [13] J. M. Charap and M. J. Duff, “Gravitational effects on Yang-Mills topology,” *Phys. Lett.* **69B** (1977) 445.
- [14] Y. Brihaye and E. Radu, “On d=4 Yang-Mills instantons in a spherically symmetric background,” *Europhys. Lett.* **75** (2006) 730 [hep-th/0605111].
- [15] E. Radu, D. H. Tchrakian and Y. Yang, “Spherically symmetric selfdual Yang-Mills instantons on curved backgrounds in all even dimensions,” *Phys. Rev. D* **77** (2008) 044017 [arXiv:0707.1270 [hep-th]].
- [16] J. D. Bekenstein, “Black hole hair: 25 - years after,” in “Moscow 1996, 2nd International A.D. Sakharov Conference on physics”, pp. 216-219 [gr-qc/9605059].
- [17] M. S. Volkov and D. V. Gal'tsov, “Gravitating non-Abelian solitons and black holes with Yang-Mills fields,” *Phys. Rept.* **319** (1999) 1 [hep-th/9810070].
- [18] C. A. R. Herdeiro and E. Radu, “Asymptotically flat black holes with scalar hair: a review,” *Int. J. Mod. Phys. D* **24** (2015) no.09, 1542014 [arXiv:1504.08209 [gr-qc]].
- [19] S. Deser, R. Jackiw and S. Templeton, “Three-dimensional massive gauge theories,” *Phys. Rev. Lett.* **48** (1982) 975.
- [20] S. Deser, R. Jackiw and S. Templeton, “Topologically massive gauge theories,” *Annals Phys.* **140** (1982) 372 [*Annals Phys.* **281** (2000) 409] Erratum: [*Annals Phys.* **185** (1988) 406].
- [21] R. Jackiw and S. Y. Pi, “Chern-Simons modification of general relativity,” *Phys. Rev. D* **68** (2003) 104012 [gr-qc/0308071].